

### ↳ The degree sequence of the graph (e.g.)

- **Example 4:** Prove that there does not exist a polyhedron with an odd number of faces such that each face has an odd number of edges.
- ✓ Such a polyhedron does not exist.

**Prove:**

- ① Suppose there are  $n$  faces, and each face has  $a_1, a_2, \dots, a_n$  edges.
- ② If all these numbers are odd, then the sum  $a_1 + a_2 + \dots + a_n$  is an odd number.
- ③ However, each edge is shared by exactly two faces, which means each edge is counted twice in the sum above.
- ④ Therefore, if there are  $m$  edges in total, then  $2m = a_1 + a_2 + \dots + a_n$ , this leads to an even number equaling an odd number – a **contradiction**.

### ↳ The degree sequence of the graph (e.g.)

- **Example 5:** Suppose a simple undirected graph of order 9 has each vertex of degree 5 or 6. Prove that it has at least 5 vertices of degree 6 or at least 6 vertices of degree 5.

**Proof 1:** Consider all possible cases. Suppose there are  $a$  vertices of degree 5 and  $b$  vertices of degree 6.

(1)  $a=0, b=9$ ;

(2)  $a=2, b=7$ ;

(3)  $a=4, b=5$ ;

(4)  $a=6, b=3$ ;

(5)  $a=8, b=1$

(1)–(3): At least 5 vertices of degree 6

(4) and (5): At least 6 vertices of degree 5

**Proof 2:** Suppose  $b < 5$ , then  $a > 9 - 5 = 4$ .  
By a corollary of the Handshaking Lemma,  $a \geq 6$ .

### ↳ 6.1 Basic Concepts of Graphs

- 6.1.1 Undirected and Directed Graphs
- 6.1.2 Vertex Degree and the Handshaking Lemma
- 6.1.3 Common Types of Graphs
- 6.1.4 Subgraphs and Complements
- 6.1.5 Graph Isomorphism

## ↳ Multigraphs and Simple Graphs

## ■ Definition 6.3:

- (1) In an *undirected graph*, two or more edges connecting the same pair of vertices are called *parallel edges*, and the number of such edges is called the **multiplicity**.
- (2) In a *directed graph*, two or more edges with the same starting and ending vertices are called *directed parallel edges* (or simply **parallel edges**), and their number is also referred to as the **multiplicity**.
- (3) A graph that contains parallel edges is called a *multigraph*.
- (4) A graph that has neither parallel edges nor loops is called a *simple graph*.

## ↳ Determining a Simple Graph (e.g.)

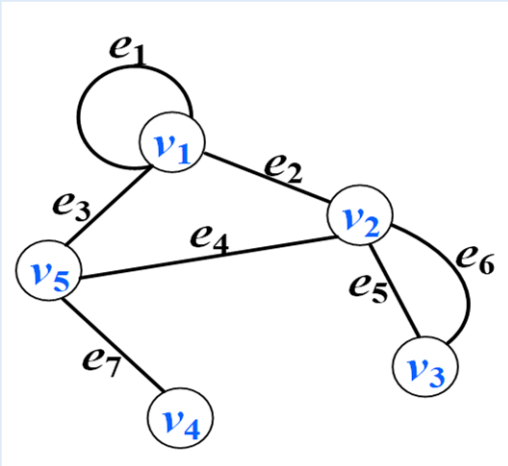


Figure 1

- Edges  $e_5$  and  $e_6$  are parallel edges.
- The multiplicity is 2.
- It is **not a simple graph**.

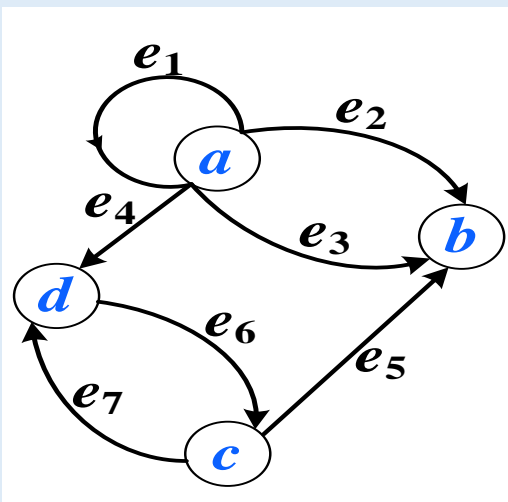
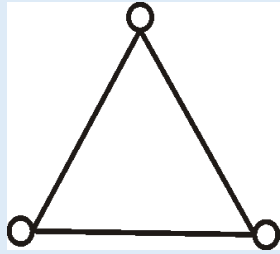


Figure 2

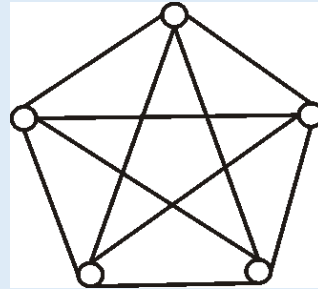
- Edges  $e_2$  and  $e_3$  are parallel edges, with a multiplicity of 2.
- Edges  $e_6$  and  $e_7$  are not parallel edges.
- It is **not a simple graph**.

## ↳ Complete Graphs and Regular Graphs

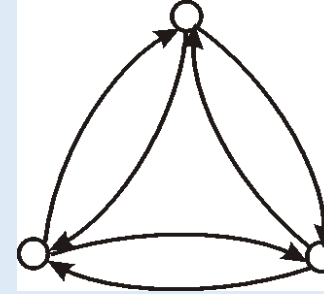
- **Undirected complete graph**: An undirected simple graph in which every pair of vertices is connected by an edge.
  - An undirected complete graph of order  $n$  is denoted by  $K_n$ , Number of vertices:  $n$ , Number of edge:  $m=n(n-1)/2$ ,  $\Delta=\delta=n-1$ .
- **Directed complete graph**: A directed simple graph in which every pair of vertices is connected by two edges in opposite directions.
  - Number of vertices:  $n$ , Number of edges:  $m=n(n-1)$ ,  $\Delta^+=\delta^+=\Delta^-=\delta^-=n-1$ ,  $\Delta=\delta=2(n-1)$ .
- **$k$ -regular graph**: An undirected simple graph in which every vertex has degree  $k$ .
  - Number of vertices:  $n$ , Number of edges:  $m=kn/2$ .



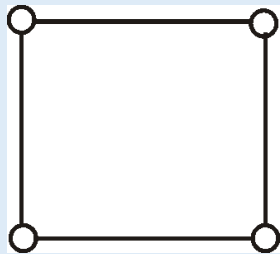
$K_3$



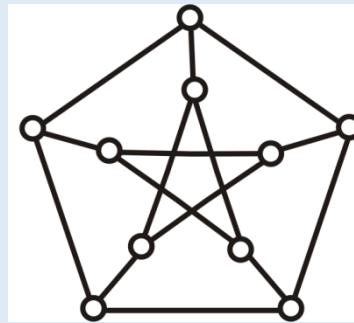
$K_5$



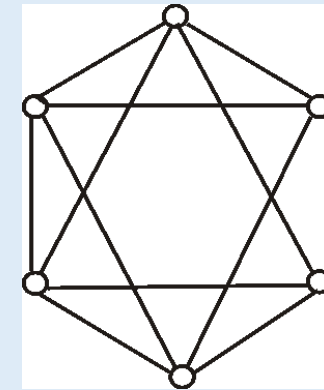
3-vertex directed  
complete graph



2-regular  
undirected graph



3-regular graph  
Petersen graph

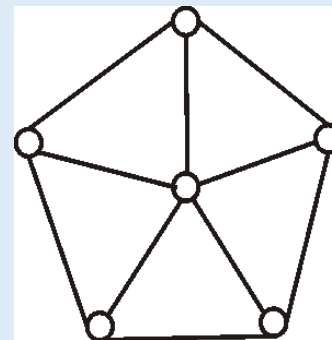
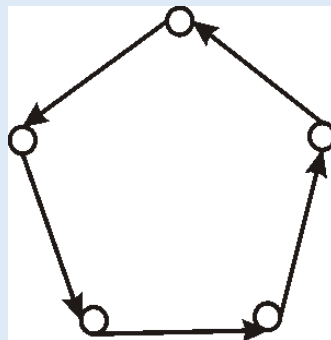
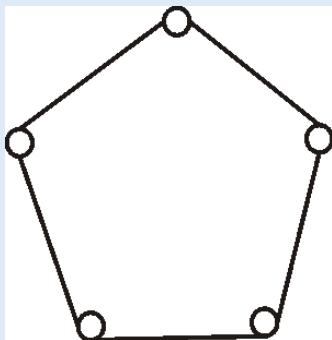


4-regular  
undirected graph

## ↳ Cycle graph and Wheel graph

- **Undirected cycle graph**  $C_n = \langle V, E \rangle$ , where  $V = \{v_1, v_2, \dots, v_n\}$ ,  $E = \{(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1)\}$ ,  $n \geq 3$ ,  $E$  is a set of unordered pairs.
- **Directed cycle graph**  $D_n = \langle V, E \rangle$ , where  $V = \{v_1, v_2, \dots, v_n\}$ ,  $E = \{\langle v_1, v_2 \rangle, \langle v_2, v_3 \rangle, \dots, \langle v_{n-1}, v_n \rangle, \langle v_n, v_1 \rangle\}$ ,  $n \geq 3$ ,  $E$  is a set of ordered pairs.
- **Wheel graph**  $W_n$ : add a single vertex to the undirected cycle graph  $C_{n-1}$  and connect it to every vertex of the cycle with exactly one edge,  $n \geq 4$ .

## ■ Example:



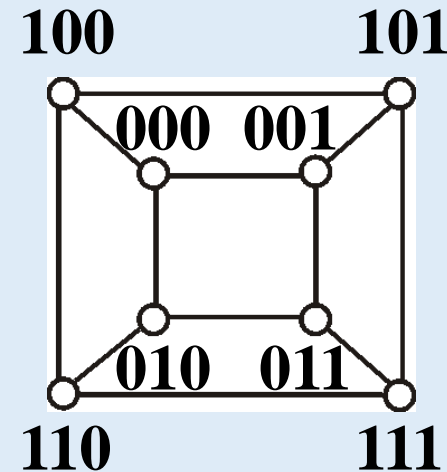
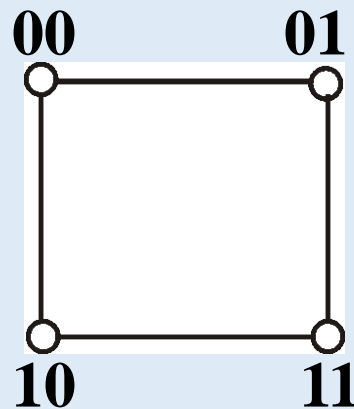
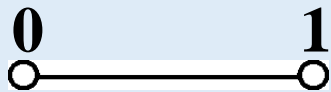
## ↳ Cycle graph and Wheel graph

- The  *$n$ -dimensional hypercube graph*  $Q_n = \langle V, E \rangle$  is a simple undirected graph of order  $2^n$ , where

$$V = \{v \mid v = a_1 a_2 \dots a_n, a_i = 0, 1, i = 1, 2, \dots, n\}$$

$$E = \{(u, v) \mid u, v \in V \wedge u \text{ and } v \text{ differ in exactly one bit position}\}.$$

- Example:



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